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Inertial and topological effects on a 2D electron gas

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Abstract

In this work, we study how the combination of rotation and a topological defect can influence the energy spectrum of a two dimensional electron gas in a strong perpendicular magnetic field. A deviation from the linear behavior of the energy as a function of magnetic field, caused by a tripartite term of the Hamiltonian, involving magnetic field, the topological charge of the defect and the rotation frequency, leads to novel features which include a range of magnetic field without corresponding Landau levels and changes in the Hall quantization steps.

1. Introduction

When a two-dimensional electron gas (2DEG) is submitted to a perpendicular magnetic field, the free electron energy levels become quantized into degenerate Landau levels (LL) [1]. This degeneracy, as well as the splitting of the levels, increases linearly with the magnetic field. Disorder broadens the levels into bands with extended states in the central region and localized states in the tail regions. This leads to a plethora of phenomena, among them, the integer quantum Hall effect (IQHE) [2], which may be affected by geometric and inertial factors. For instance, curvature is known to affect the IQHE [3–5]. In fact, curvature may be used as a tool to manipulate the electronic structure of charge carriers confined to a surface [6–8]. Rotation, as well, has its effects on the IQHE as we reported in a previous publication [9]. The aim of this article is to further elucidate the influence of curvature and rotation on the LL and on the IQHE. This is done with the inclusion of a topological defect (disclination) and by making the system rotate around the defect axis. Separately, the disclination [10] and the rotation [9, 11] introduce subtle modifications in the LL spectrum without changing the linear dependence of the energy on the magnetic field. Together, they make the energy dependence on the magnetic field become parabolic and there appears a range of magnetic field free of LL, as presented below.

There is some similarity between the application of a magnetic field and rotation on free charged particles [12, 13]. It comes mainly from the Coriolis force which, if it could act alone, would work like the magnetic force, creating analog LL. However, its companion, the centrifugal force, breaks the degeneracy and the sample-length independence of the LL. In fact, the Coriolis and centrifugal contributions to the quantum Hamiltonian together lead to a coupling between the particle total angular momentum and the rotation. Therefore, the combination of magnetic field and rotation provokes a shift and a splitting of the original LL due to this coupling [11].

We consider a non-interacting free electron gas in a rotating disk with a disclination at its rotation axis, under the influence of a perpendicular, uniform, magnetic field. Charged particles in a rotating Hall sample with a magnetic field were already studied in [11], where it was pointed out that rotation breaks the degeneracy of the LL. Furthermore, the counting of states fully occupied below the Fermi energy may change, altering the Hall quantization steps. Charged particles in a sample with a single disclination, in the presence of an orthogonal constant magnetic field, were also studied in [10], where it was found a modification in the Hall conductivity steps due to the defect. Our purpose here is to investigate the combined influence of rotation and disclination in the Hall conductivity. As expected, and in fact verified in [10, 11], we find that rotation and the disclination,

separately couple to angular momentum, as does the magnetic field. But, when rotation, disclination, and magnetic field act together on the free electron gas, a new coupling is found involving all of them simultaneously. This makes the LL as function of the magnetic field to bend from the usual straight lines and introduces a region of magnetic field without LL, as shown below.

2. Energy levels

In three dimensions, a disclination is a topological defect [14] associated to the removal of a wedge of material with the subsequent identification of the loose ends (Volterra process), introducing an angular deficit which changes the boundary condition on the angular variable from $\phi \rightarrow \phi + 2\pi$ into $\phi \rightarrow \phi + 2\pi\alpha$. Here, $\alpha < 1$ expresses the removed wedge angle of $2\pi(1 - \alpha)$. Conversely, if a wedge is added, $\alpha > 1$. Effectively, the new boundary condition can be applied by working in a background space with the line element

$$ds^2 = dr^2 + \alpha^2 r^2 d\phi^2 + dz^2. \tag{1}$$

As shown in [14], the Frank vector which characterizes a disclination is nothing more than the curvature flux associated to the defect. Since the above line element corresponds to a curvature scalar given by $R = 2\left(\frac{1-\alpha}{\alpha}\right)\frac{\delta(\rho)}{\rho}$, as shown in [15], its flux is therefore

$$\oint R\rho \, d\rho \, d\phi = 4\pi\left(\frac{1-\alpha}{\alpha}\right) = F, \tag{2}$$

giving the Frank vector F, or topological charge of the disclination. This result still holds for a 2D surface with a disclination, which is the subject of this article.

The disclination introduces anisotropy which is manifest in the effective mass approximation. In this approach, the bottom of the conduction band of a isotropic material is approximated by a parabola corresponding to the dispersion relation $E(\vec{k}) = \frac{\hbar^2}{2m^*}k^2$, where m^* is the effective mass. For anisotropic materials, this becomes $E(\vec{k}) = \sum_{i,j}\frac{\hbar^2}{2m^*_{ij}}k_i k_j$. The introduction of a disclination into an otherwise isotropic material breaks the global rotational symmetry of the system, reducing it to rotations around the disclination axis. The immediate consequence of this is the transformation of the inverse isotropic effective mass from a scalar to a tensor:

$$\frac{1}{m^*} = \begin{pmatrix} 1/m_r & 0 & 0 \\ 0 & 1/m_\phi & 0 \\ 0 & 0 & 1/m_z \end{pmatrix}, \tag{3}$$

where (r, ϕ, z) are the usual cylindrical coordinates. The kinetic energy operator then becomes

$$K = -\frac{\hbar^2}{2}\left[\frac{1}{m_r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{1}{m_\phi r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{m_z}\frac{\partial^2}{\partial z^2}\right] \tag{4}$$

which, for a 2D system can be written as

$$K = -\frac{\hbar^2}{2m_r}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{(\alpha')^2 r^2}\frac{\partial^2}{\partial \phi^2}\right], \tag{5}$$

where $\alpha' = m_\phi/m_r$. Of course α and α' are not necessarily the same but the existence of the former implies in the appearance of the latter in the kinetic energy operator. The term between square brackets is the Laplacian in the effective geometry associated to the line element given by equation (1) with α replaced by α'. This being clear, on what follows, we drop the prime from α' and the subscript r from m_r for simplicity of notation.

As discussed in [11], the Hamiltonian in cylindrical coordinates of a particle in a rotating disk, in the presence of a magnetic field $\vec{B} = B\hat{z}$, can be written as

$$H = \frac{[\vec{p} - q\vec{A} - m(\vec{\Omega} \times \vec{r})]^2}{2m} - \frac{m(\vec{\Omega} \times \vec{r})^2}{2} + qV, \tag{6}$$

where V and $\vec{A}$ are the scalar and vector electromagnetic potentials, given by

$$V = -\frac{\Omega B r^2}{2}, \tag{7}$$

$$\vec{A} = \left(0, \frac{Br}{2\alpha}, 0\right) \tag{8}$$

and $\vec{\Omega} = \Omega\hat{z}$ is the angular velocity of the disk. The electric field associated to the scalar potential appears from the transformation of the applied magnetic field to the rotating frame. The disclination factor, α, appearing in

the vector potential compensates the change $\phi \rightarrow \alpha\phi$ giving the correct value of the magnetic flux through a circle of radius r in the plane. That is, $\oint A_\phi \mathrm{d}s = \oint \frac{Br}{2\alpha} r\alpha\mathrm{d}\phi = \pi r^2 B$, where $\mathrm{d}s = r\alpha\mathrm{d}\phi$ comes from the metric (1) with $z = \mathrm{const.}$ and $r = \mathrm{const.}$, which define the circle. Again, due to the change $\phi \rightarrow \alpha\phi$, we write the angular momentum operator as $\vec{p}_\phi = -\hat{\phi}\frac{i\hbar}{\alpha r}\frac{\partial}{\partial\phi}$. Thus, the Hamiltonian can be written as

$$H = \frac{p^2}{2m} - \mu r p_\phi + \beta r^2, \tag{9}$$

with

$$\mu = \frac{qB}{2m\alpha} + \Omega \tag{10}$$

and

$$\beta = \frac{q^2 B^2}{8m\alpha^2} + \frac{qB\Omega F}{8\pi}. \tag{11}$$

The $\mu r p_\phi$ term in equation (9) contains the usual coupling between the magnetic field and the angular momentum plus the coupling between rotation and angular momentum (see equation (10)). The quadratic term in equation (9) has the usual contribution from the magnetic field and a term which couples magnetic field B, rotation speed Ω and the topological charge F of the defect (see equation (11)).

For this Hamiltonian, the Schrödinger equation can be written as

$$-\frac{\hbar^2}{2m}\nabla^2\psi + i\frac{\mu\hbar}{\alpha}\frac{\partial\psi}{\partial\phi} + \beta r^2\psi = E\psi, \tag{12}$$

where the Laplacian operator (in fact the Laplace-Beltrami operator) is given by $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{\alpha^2 r^2}\frac{\partial^2}{\partial\phi^2}$ (see equation (5)). With the ansatz $\psi = R(r)\mathrm{e}^{-\mathrm{i}l\phi}$, equation (12) becomes

$$r^2 R'' + rR' + \left(-\sigma^2 r^4 + \gamma^2 r^2 - \frac{\ell^2}{\alpha^2}\right)R = 0, \tag{13}$$

where

$$\sigma^2 = \frac{q^2 B^2}{4\hbar^2\alpha^2} + \frac{mqB\Omega}{\hbar^2}\left(\frac{1-\alpha}{\alpha}\right) \tag{14}$$

and

$$\gamma^2 = \frac{2m}{\hbar}\left(\frac{E}{\hbar} - \frac{qB\ell}{2m\alpha^2} - \frac{\Omega\ell}{\alpha}\right). \tag{15}$$

Writing $\sigma r^2 = \xi$ and looking at the asymptotic limit as $\xi \rightarrow \infty$, the general solution to this equation will be given in terms of $M(a, b, \xi)$, the *confluent hypergeometric function of the first kind* [16],

$$R \equiv R_\ell(\xi) = a_\ell \mathrm{e}^{-\frac{\xi}{2}}\xi^{\frac{|\ell|}{2\alpha}}M\left(\frac{-\gamma^2}{4\sigma} + \frac{|\ell|}{2\alpha} + \frac{1}{2}, 1 + \frac{|\ell|}{\alpha}, \xi\right)$$

$$+ b_\ell \mathrm{e}^{-\frac{\xi}{2}}\xi^{-\frac{|\ell|}{2\alpha}}M\left(\frac{-\gamma^2}{4\sigma} - \frac{|\ell|}{2\alpha} + \frac{1}{2}, 1 - \frac{|\ell|}{\alpha}, \xi\right). \tag{16}$$

In equation (16), a_ℓ and b_ℓ are, respectively, the coefficients of the *regular* and *irregular* solutions. Notice that the term *irregular* stems from the fact that it diverges as $\xi \rightarrow 0$. We consider this case since the disclination, described by a cone-like background, may introduce a singular curvature potential in the problem [17]. As showed in [18] in the study of the emission of x-ray laser beams from metals, singularities have important impact in quantum systems. In dealing with singularities, extra care is needed since, when they occur, the Hamiltonian may not be not self-adjoint. For the specific case of the conical singularity associated to a disclination [19], gives the prescription to make a Hamiltonian self-adjoint by extending its domain. This leads to the inclusion of the *irregular* term in equation (16). The *regular* solution corresponds to the case where the core of the disclination which carries the curvature singularity is artificially removed (not considered). For the sake of comparison we present our results bellow for both cases: *regular* (core removed) and *total* (core included, meaning *regular* + *irregular*).

In order to have a finite polynomial function (the hypergeometric series has to be convergent in order to have a physical solution), the condition $a = -n$, where n is a positive integer number, has to be satisfied. From this condition, the possible values for the energy are given by

$$E = \frac{\hbar \omega_c \ell}{2\alpha^2} + \frac{\hbar \Omega \ell}{\alpha}$$
$$+ \hbar \sqrt{\frac{\omega_c^2}{\alpha^2} + 4\omega_c \Omega \frac{(1-\alpha)}{\alpha}} \left[n \pm \frac{|\ell|}{2\alpha} + \frac{1}{2} \right], \qquad (17)$$

where $\omega_c = qB/m$ is the cyclotron frequency. It should be noted that, for the case where the singular interaction is absent, only the regular solutions contribute for the bound state wave function ($b_\ell \equiv 0$), and the energy is given by equation (17) with the plus sign together with the condition $|\frac{\ell}{\alpha}| \geqslant 1$. On the other hand, if a singular interaction is present, then the energy levels with the minus sign exist but the constraint $|\frac{\ell}{\alpha}| < 1$ must be observed since this is the condition for the irregular wave function to be square integrable [20]. The energy spectrum above must be analyzed in terms of the values the parameter α can assume, since the condition $|\frac{\ell}{\alpha}| \geqslant 1$ ($|\frac{\ell}{\alpha}| < 1$) for the regular (irregular) solution has to be fulfilled. In summary, in the absence of the coupling between the wavefunctions and the singular curvature, the constraint $|\frac{\ell}{\alpha}| \geqslant 1$ must be imposed, which restricts the allowed momentum eigenvalues ℓ. This guarantees that such wave functions are regular as $r \to 0$. If such coupling is present, then the particles are allowed to have other values of the angular momentum eigenvalues ℓ without the constraint. We will examine these two situations in order to show how singular effects are important when they appear in the system.

We remark that the fields B and Ω are externally tunable parameters. Now, considering $\alpha = 1$ and $\Omega \neq 0$, i.e., when the system rotates but has no disclination, we recover the result found in [11]. On the other hand, by considering $\alpha \neq 1$ and $\Omega = 0$, we recover the result presented in the [10, 21]. The disclination alone introduces a change in the angular momentum that breaks the infinite degeneracy of the LL [21]. The presence of rotation by itself breaks the degeneracy between states with opposite angular momentum and introduces an energy shift depending on the rotation [11]. But, when we have magnetic field, disclination and rotation simultaneously, there exists a term in the energy, $4\omega_c \Omega \frac{(1-\alpha)}{\alpha}$, that represents the coupling of these three elements, i.e., if either $\Omega = 0$, $\alpha = 1$ or $B = 0$ this term vanishes from the energy expression.

In the absence of magnetic field, the nature of the wave function is completely changed as seen below. Considering $B = 0$, equation (13) becomes

$$r^2 R'' + r R' + \left(\gamma^2 r^2 - \frac{\ell^2}{\alpha^2} \right) R = 0, \qquad (18)$$

which is a Bessel equation with $\gamma^2 = \frac{2m}{\hbar} \left(\frac{E}{\hbar} - \frac{\Omega l}{\alpha} \right)$. Then the energy spectrum can be written as

$$E = \frac{\hbar^2 \gamma^2}{2m} + \frac{\hbar \Omega \ell}{\alpha} \qquad (19)$$
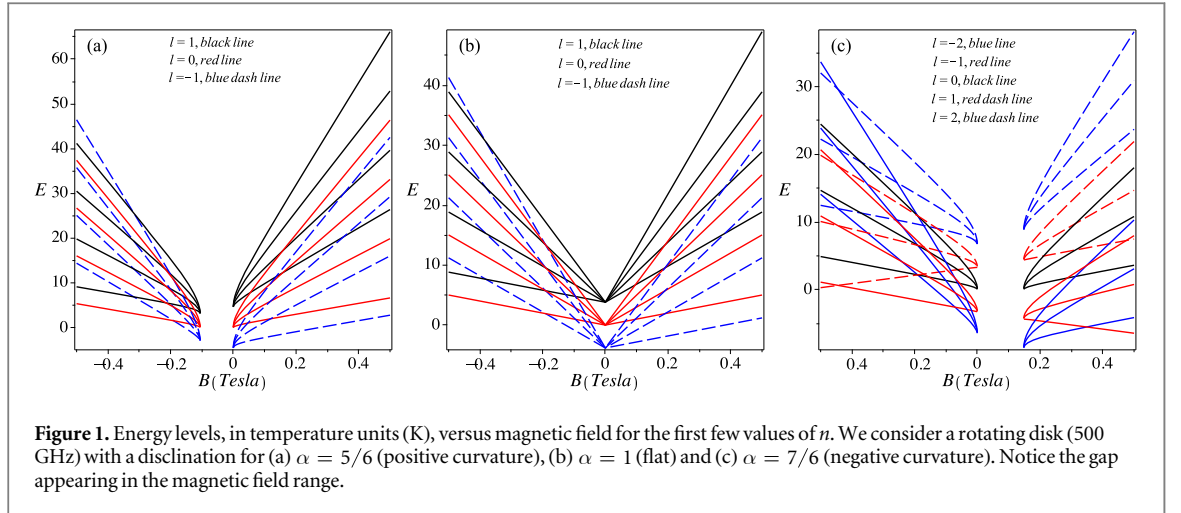
and the wave function is

$$\psi = J_{\ell/\alpha}(|\gamma|r) e^{-\frac{i\ell\theta}{\alpha}}, \qquad (20)$$

where γ is a continuous variable if the sample is infinite. For a finite size sample, γ is discrete and is found by the condition $J_{\ell/\alpha}(|\gamma|a) = 0$, where a is the radius of the sample. Therefore, the presence of the disclination modifies the result obtained by Johnson in [12], who studied the effect of rotation on the energy spectrum of free charges and its consequences to the Hall effect. Effectively, we obtain a rescaling of the angular momentum ℓ by the defect parameter α as seen in equations (19)–(20). For $\alpha = 1$ we recover the result of Johnson [12].

3. Electronic structure

The energy levels of a rotating disk with a disclination in the presence of a magnetic field are given by equation (17), whose representation can be seen in figure 1. We used $\alpha = 5/6$ and $\alpha = 7/6$, realistic values for the hexagonal lattice, which correspond to positive and negative curvature defects, respectively. The values of the quantum number ℓ, considering the irregular part of the spectrum, are indicated in the figures. As expected [11], the presence of rotation splits the levels. When there is also a disclination in the system, there appears a range of magnetic field without corresponding bound energy states (LL). This happens when the square root in equation (17) becomes imaginary for $\frac{\omega_c^2}{\alpha^2} + 4\omega_c \Omega \frac{(1-\alpha)}{\alpha} < 0$, which may be obtained by either inverting the direction of rotation or the magnetic field. Or else, by replacing a sample with a positive curvature disclination ($\alpha < 1$) by another one with a negative curvature disclination ($\alpha > 1$), which is what we present in figure 1.

The term $4\omega_c \Omega \frac{(1-\alpha)}{\alpha}$ exists only when magnetic field, rotation and a disclination occur simultaneously in the sample. This explains why there is no bound state (LL) in the magnetic field range $-\frac{4m\Omega}{q}\alpha(1-\alpha) < B < 0$ if $\alpha < 1$ or $0 < B < \frac{4m\Omega}{q}\alpha(\alpha - 1)$ if $\alpha > 1$. Notice that this range depends on the material through the

Figure 1. Energy levels, in temperature units (K), versus magnetic field for the first few values of *n*. We consider a rotating disk (500 GHz) with a disclination for (a) $\alpha = 5/6$ (positive curvature), (b) $\alpha = 1$ (flat) and (c) $\alpha = 7/6$ (negative curvature). Notice the gap appearing in the magnetic field range.

effective electron mass, *m*. In summary, there is a competition between the three elements (B, Ω and α) which affects directly the energy spectrum. The scattering states for *B* in the aforementioned range is due to the fact that, in this range, the magnetic field is not strong enough to overcome the combined effect of rotation and disclination. Since the gap in the magnetic field range, shown in figure 1, destroys the LL, rotation (plus disclination) can be used to bring the Hall effect to the classical regime, where many applications as Hall sensors abound.

Without the term $4\omega_c\Omega\frac{(1-\alpha)}{\alpha}$, the energy is linear in *B* as easily seen in equation (17). The presence of this term distorts the straight lines into the parabolas shown in figure 1. Also, in figure 1, it is clear the splitting of levels due to rotation, and how the disclination mediates this effect.

4. Hall conductivity

At zero temperature and considering that the Fermi energy E_F is in an energy gap, the Hall conductivity can be written as [22]

$$\sigma_H(E_F, 0) = \frac{e}{S}\frac{\partial N}{\partial B}, \tag{21}$$

where *N* is the number of states below the Fermi energy and *S* is the area of the surface. The density of states is given by

$$n(E) = \frac{|eB|}{2\pi\hbar}\sum_{n,l}\delta(E - E_{n,l}). \tag{22}$$

Therefore, one can obtain *N*, as

$$N = S\int_{-\infty}^{E_F} n(E)\,dE = \frac{S|eB|}{h}n_0, \tag{23}$$

where $n_0 = (n + 1)n_\ell$ is the number of fully occupied LLs below E_F. The number of occupied ℓ states is represented by n_ℓ. The Hall conductivity is then
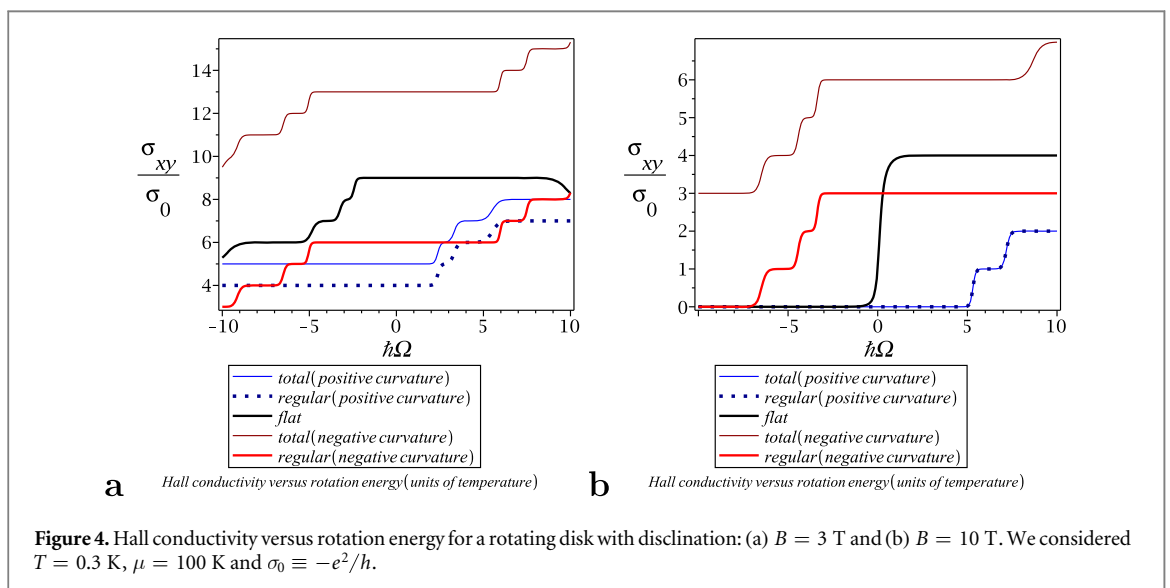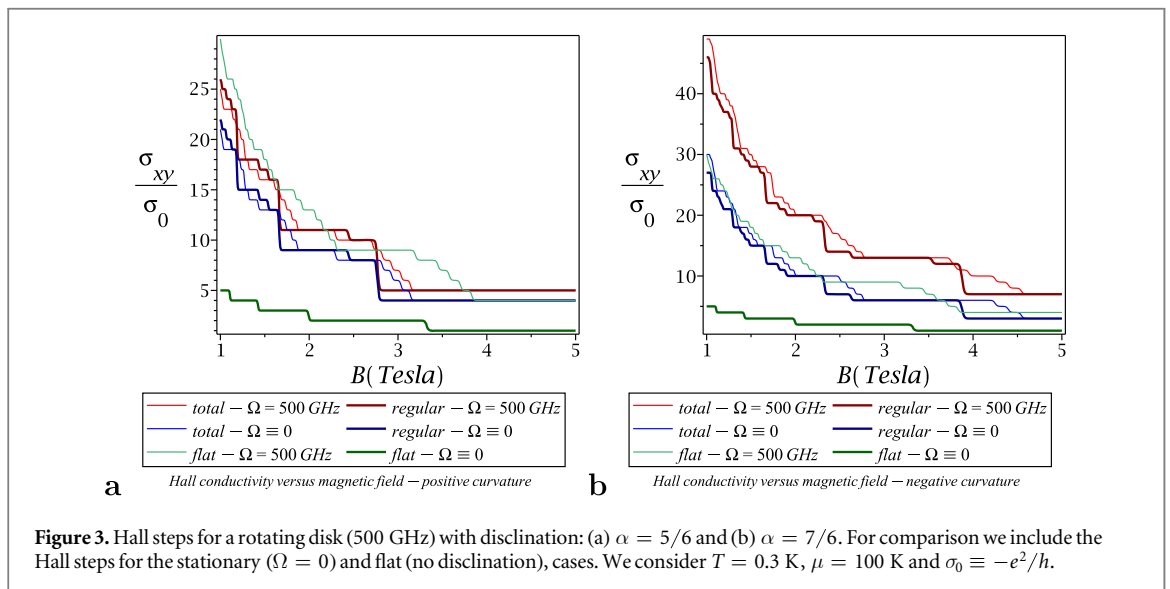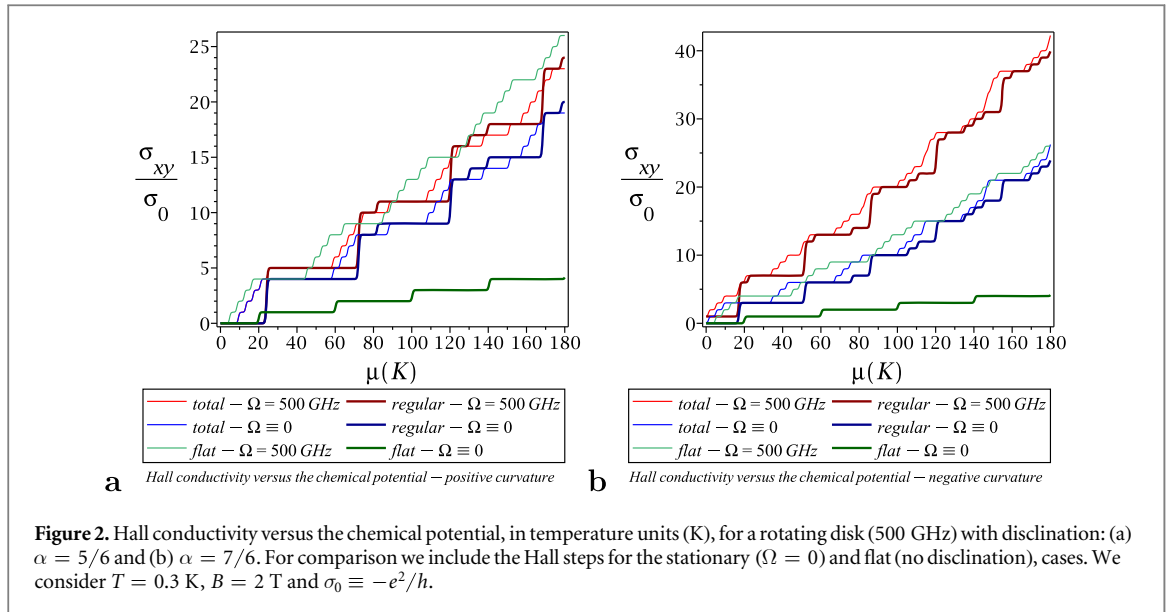
$$\sigma_H(E_F, 0) = -\frac{e^2}{h}n_o. \tag{24}$$

For $T \neq 0$, we consider the expression for the Hall conductivity obtained in the clean limit (absence of impurities) given by [23]

$$\sigma_H(\mu, T) = \int_{-\infty}^{\infty}\left(-\frac{\partial f_0}{\partial E}\right)\sigma_H(E, 0)\,dE, \tag{25}$$

where μ is the chemical potential, *T* is the temperature and f_0 is the Fermi–Dirac distribution. We express the energy scale in units of temperature. We consider the parameters of a GaAS: $m = 0,067m_e$, where $m_e = 9,11 \times 10^{-28}$ g is the electron rest mass.

The Hall conductivity versus the chemical potential is depicted in figure 2, versus the magnetic field in figure 3 and versus the rotation speed in figure 4. We recall that *total* corresponds to the case where the effect of the defect singularity was taken into account and *regular* if this is not considered. As the magnetic field is increased while keeping the chemical potential constant, the energy levels move upwards, which is equivalent to

a   *Hall conductivity versus the chemical potential — positive curvature*

b   *Hall conductivity versus the chemical potential — negative curvature*

Figure 2. Hall conductivity versus the chemical potential, in temperature units (K), for a rotating disk (500 GHz) with disclination: (a) $\alpha = 5/6$ and (b) $\alpha = 7/6$. For comparison we include the Hall steps for the stationary ($\Omega = 0$) and flat (no disclination), cases. We consider $T = 0.3$ K, $B = 2$ T and $\sigma_0 \equiv -e^2/h$.

a   *Hall conductivity versus magnetic field — positive curvature*

b   *Hall conductivity versus magnetic field — negative curvature*

Figure 3. Hall steps for a rotating disk (500 GHz) with disclination: (a) $\alpha = 5/6$ and (b) $\alpha = 7/6$. For comparison we include the Hall steps for the stationary ($\Omega = 0$) and flat (no disclination), cases. We consider $T = 0.3$ K, $\mu = 100$ K and $\sigma_0 \equiv -e^2/h$.

a   *Hall conductivity versus rotation energy (units of temperature)*

b   *Hall conductivity versus rotation energy (units of temperature)*

Figure 4. Hall conductivity versus rotation energy for a rotating disk with disclination: (a) $B = 3$ T and (b) $B = 10$ T. We considered $T = 0.3$ K, $\mu = 100$ K and $\sigma_0 \equiv -e^2/h$.

make the highest occupied state move down. This leads to a reduction of the Hall conductivity as shown in figure 3. Considering a rotation speed of 500 GHz, for either positive ($\alpha = 5/6$) or negative curvature ($\alpha = 7/6$) disclination, the Hall conductivity is enhanced in the interval for the chemical potential shown in figure 2. The same occurs in the plots of the Hall conductivity versus the magnetic field shown in figure 3. Notice also that the singularity has significant impact in the formation of the Hall steps as evidenced by the *total* versus *regular* curves. Figures 4(a) and (b) show the formation of plateaus due to the rotation, for different values of the magnetic field.

5. Conclusion

In this paper, we investigated the influence of rotation along with a disclination on the quantized Hall conductivity of a 2DEG. For this purpose, we considered a rotating non-interacting planar 2DEG in the presence of a disclination and a perpendicular uniform magnetic field. We found that, besides the joint influence of magnetic field and rotation, magnetic field and disclination, and rotation and disclination, observed in previous works, there is a simultaneous coupling between rotation, magnetic field and disclination. This interaction gives rise to a range of values for the magnetic field that does not bind the electrons into LL. The size of this region depends on the effective mass of the charge carrier, the disclination and/or the rotation rate. The most important feature of this effect is that it only exists when the three elements (magnetic field, disclination and rotation) are present simultaneously. This effect will be noted for low magnetic fields, where the Drude model can be employed to analyze the Hall effect. This means that such effect can find applications in the context of Hall sensors, for instance. In a previous work [11], we verified that the rotation breaks the degeneracy of the LL. Here, we verified that the disclination can leverage the splitting of degenerated states. We also found the Hall conductivity for this system, which is similar to the case without rotation and disclination. Nevertheless, due to the splitting of energy levels caused by rotation and enhanced by the disclination, the number of states under the Fermi level increases, raising the Hall conductivity with respect to the previous study. As a perspective, we propose to analyze the influence of rotation and disclination together in the quantum Hall effect of novel materials such as graphene and topological insulators. Also of interest is the investigation of the influence of rotation and disclination in related phenomena like de Haas–van Alphen effect.

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