



Fermionic wave functions and Grassmann fields as possible sources of dark energy

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Abstract We study a cosmological model with a fermionic field which can be interpreted as a source of dark energy in the universe. Two different approaches were considered, the first one with a massless fermionic field represented by a standard wave-function and the second one where a massive field is a Grassmann variable. The first case naturally reduces to a Λ CDM model with a constant equation of state parameter, while the last case reproduces a $w(z)$ CDM model for a massive field, and in the massless limit, the intrinsic Grassmannian property of the field leads always to a vacuum equation of state parameter, irrespective the specific form of the potential. Both cases lead to a dark energy contribution of the fermionic sector. The models are totally compatible with recent cosmological data from Supernovae, BAO and Hubble parameter measurements. A brief study of linear evolution of density perturbations shows that some of the small scale problems related to standard model can be at least alleviated.

1 Introduction

The present status of the astronomical observations of the Cosmic Microwave Background radiation, the large-scale

distribution of galaxies, and distant Supernovae of Type Ia (SNe Ia), among others, lead us to an accurate cosmic concordance model for cosmology [1]. The model incorporates, beyond the standard baryonic matter, a sector described by the cosmological constant Λ (or dark energy) and the Cold Dark Matter (CDM), and is called the Λ CDM model.

The dark sector is a fundamental part of the model, since they represent about 95% of the total material content of the Universe, what allows the description of astronomical data accurately. On the other hand, it also brought to cosmology some of the most challenging problems of modern science. In particular, the knowledge of the nature of dark energy and dark matter is far from being settled [2,3].

In the present work we review a model where the dark energy component in the cosmic evolution is a single fermionic field, permitting an alternative description of the dark energy sector of the Λ CDM. Our purpose is to discuss theoretical aspects that, as we will see, may have cosmological consequences. The work, we believe, adds new knowledge to the numerical analysis presented in other papers [4,5]. Besides, we will highlight to what extent the model agrees with recent astronomical data [6–9].

For a starting motivation for introducing fermions in cosmology, we may look for the history of analogies in particle physics. The first unified description of states of nucleons and mesons as composed states of a fundamental fermion, for example, was introduced in the seminal papers by Nambu and Jona-Lasinio in analogy with superconductivity [10,11]. Beyond analogy, we know now that we must interpret the

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Nambu–Jona-Lasinio model as an effective quantum field theory, valid at sufficiently low energy [12]. At present, fermions are well established as a fundamental part of the description of nature, representing quarks and leptons in the present Standard Model (SM) of elementary particles [13].

However, although we know now that the SM is the theory underlying an effective model like that proposed by Nambu and Jona-Lasinio, we do not know what fundamental theory is beyond the Λ CDM model [14]. Thus, a description based on an analogy with the fields of the SM opens the possibility to explore not only the fermion component but also the scalar [15] and vector [16] degrees of freedom. A possible vector component does not concern us here because we consider an isotropic universe. But it is interesting to make some comments about models based on scalar field components and how they may be related to fermions. Again, we look for insights into particle physics and quantum field theory.

It is well established that the dynamic of the SM is in a particular representation of the vacuum where the electroweak symmetry $SU(2) \times U(1)$ is spontaneously broken to $U(1)$. The process which explains the breaking of the gauge symmetry of the model is a second-order phase transition occurring in a quantum field theory at finite temperature. This picture is introduced in the model by the Higgs sector, described by a scalar field that in the broken phase which we observe at present gives mass to the leptons and the right massive modes correspondent to the weak vector bosons and the massless photon of the electromagnetism. Of course, it is important for cosmology since this mechanism furnish a natural description of the early universe [17].

Although the scalar sector of the SM introduces problems to the dynamics that are undesirable from the physical point of view [18]. Among the ideas proposed to solve or at least give a better understanding of these problems, we find the interpretation of the Higgs field as a composite of fermion fields that extend the degrees of freedom of the dynamics beyond the Standard Model [19–21].

Recent works have studied the cosmological implications of fermionic fields in different contexts, as inflation, dark matter and dark energy [4, 5, 22–26]. The purpose of the present paper is to explore a possible analogy between particle physics and cosmology based on the idea that we can interpret the dark energy sector in terms of a fundamental fermion. In particular, we analyze the cosmological consequences of treating the fermion as a single wave-function or a Grassmann classical field.

The paper is organized as follows. The basic equations of the model are presented in Sect. 2. In Sect. 3, we discuss the cases in which the fermion is treated as a wave function or Grassmann field. We consider the constraint of the models with observational data in Sect. 4. A brief study of linear evolution of density perturbations is presented in Sect. 5. In Sect. 6, we present the conclusions.

2 Fermions coupled to gravity

We start with the Lagrangian density of the model of Dirac fermions¹ ψ :

$$\mathcal{L}_f = \frac{i}{2} [\bar{\psi} \Gamma^\mu D_\mu \psi - (D_\mu \bar{\psi}) \Gamma^\mu \psi] - m \bar{\psi} \psi - V(\bar{\psi}, \psi), \quad (1)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, $\Gamma^\mu = e_a^\mu \gamma^a$, m is a constant with dimension of mass, and $V(\bar{\psi}, \psi)$ is the potential with self-interactions constructed from the bilinears $\bar{\psi} \psi$ and $i \bar{\psi} \gamma_5 \psi$. The covariant derivative and the spin connection are [4]

$$D_\mu \psi = \partial_\mu \psi - \Omega_\mu \psi, \quad (2)$$

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \bar{\psi} \Omega_\mu, \quad (3)$$

$$\Omega_\mu = -\frac{1}{4} g_{\rho\sigma} [\Gamma_{\mu\nu}^\rho - e_a^\rho \partial_\mu e_\nu^a] \Gamma^\sigma \Gamma^\nu. \quad (4)$$

The γ^a are the Dirac matrices, e_a^μ are the set of tetrad defined at every point of the gravitational field, and we also have $\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

The equations of motion of the fermion in the presence of the gravitational field are

$$\begin{aligned} i \Gamma^\mu D_\mu \psi - m \psi - \frac{dV}{d\bar{\psi}} &= 0 \\ i D_\mu \bar{\psi} \Gamma^\mu + m \bar{\psi} + \frac{dV}{d\psi} &= 0. \end{aligned} \quad (5)$$

The Einstein equations of the metric $g_{\mu\nu}$ are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}, \quad (6)$$

where $T_{\mu\nu} = T_{m\mu\nu} + T_{f\mu\nu}$ is the total energy-momentum tensor, with $T_{m\mu\nu}$ representing the energy-momentum tensor for matter and $T_{f\mu\nu}$ for the fermion field ψ , obtained by:

$$\delta S_f = -\frac{1}{2} \int d^4x \sqrt{-g} T_f^{\mu\nu} \delta g_{\mu\nu}. \quad (7)$$

Using the Lagrangian density (1), the variation of the action S_f with relation to the metric $g_{\mu\nu}$ gives:

$$\begin{aligned} T_f^{\mu\nu} = \frac{i}{4} [\bar{\psi} \Gamma^\nu D^\mu \psi + \bar{\psi} \Gamma^\mu D^\nu \psi \\ - D^\mu \bar{\psi} \Gamma^\nu \psi - D^\nu \bar{\psi} \Gamma^\mu \psi] - g^{\mu\nu} \mathcal{L}_f. \end{aligned} \quad (8)$$

From this, we have the energy density ρ_f and pressure p_f of the fermion considered as a perfect fluid:

¹ In natural units.

$$T_{f\ \nu}^{\mu} = g_{\nu\sigma} T_f^{\mu\sigma} = (\rho_f, -p_f, -p_f, -p_f). \quad (9)$$

The expressions for pressure and energy density comes from the expression (8). As a consequence of the homogeneity of the space, we have that ψ is a function only of time. It furnishes simplified expressions for the covariant derivative:

$$D_i \psi = -\Omega_i \psi \quad D_i \bar{\psi} = \bar{\psi} \Omega_i \quad (10)$$

and

$$D_0 \psi = \dot{\psi} \quad D_0 \bar{\psi} = \dot{\bar{\psi}}. \quad (11)$$

Here, the “dot” on the ψ and $\bar{\psi}$ represents derivative with relation to time, and we have used the fact that $\Omega_0 = 0$. Since

$$\Gamma^0 = \gamma^0, \quad \Gamma^i = \frac{1}{a(t)} \gamma^i \quad \text{and} \quad \Omega_i = \frac{1}{2} \dot{a}(t) \gamma^i \gamma^0, \quad (12)$$

we have

$$\Gamma^i \Omega_j + \Omega_j \Gamma^i = 0. \quad (13)$$

Using this identity and the equations of motion (5) in the expression (8) for the energy–momentum tensor we get

$$p_f = \frac{1}{2} \bar{\psi} \frac{dV}{d\bar{\psi}} + \frac{1}{2} \frac{dV}{d\psi} \psi - V(\bar{\psi}, \psi) \quad (14)$$

and

$$\rho_f = m \bar{\psi} \psi + V(\bar{\psi}, \psi). \quad (15)$$

Taking the covariant differentiation of Einstein field equations (6) we obtain the conservation of the total energy–momentum tensor:

$$\dot{\rho} + 3H(\rho + p + \bar{\omega}) = 0 \quad (16)$$

where $H = \dot{a}/a$ and $\bar{\omega}$ represents the nonequilibrium pressure when dissipative processes are taken into account [27–30]. In which follows we will assume $\bar{\omega} \approx 0$.

3 Wave function versus Grassmann variables

The terms with $\frac{dV}{d\bar{\psi}}$ and $\frac{dV}{d\psi}$ which appear in the expression (14) may have interesting implications for the model. It is because the result for the derivatives depends if we treat ψ as a wave function or a relativistic classical field. In the last

case, the fermion statistic demands the field to be a Grassmann variable. We can consider both cases in the calculations observing that

$$\frac{d(\bar{\psi}\psi)}{d\psi} = s\bar{\psi} \quad \text{and} \quad \frac{d(\bar{\psi}\gamma^5\psi)}{d\psi} = s\bar{\psi}\gamma^5, \quad (17)$$

where $s = 1$ corresponds to the ψ treated as a wave function, and $s = -1$ to the case that it is a classical field described by a Grassmann variable.

As pointed in [4], the most general form for the self-interaction potential $V(\bar{\psi}, \psi)$ constructed from the Lorentz invariant bilinears $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^5\psi$ is

$$V(\bar{\psi}, \psi) = \left[\beta_1 (\bar{\psi}\psi)^2 + \beta_2 (i\bar{\psi}\gamma^5\psi)^2 \right]^n, \quad (18)$$

where n , β_1 and β_2 are constants. It is interesting to note that this potential introduce natural scales in the problem since the coupling constants β_1 and β_2 have dimension $2(3n - 2)$ in units of mass (in four dimension). It may be of practical use in the classical theories and, more important, will have fundamental implications in a quantum field theory.

Let us move back to the problem on we are interested and calculate the pressure. Using the derivatives (17) we see that

$$\frac{1}{2} \bar{\psi} \frac{dV}{d\bar{\psi}} = n V(\bar{\psi}, \psi) \quad (19)$$

and

$$\frac{1}{2} \frac{dV}{d\psi} \psi = s n V(\bar{\psi}, \psi). \quad (20)$$

Now, from the Eq. (14) we have

$$p_f = [(1 + s)n - 1] V(\bar{\psi}, \psi). \quad (21)$$

Now, it is convenient to separate the analysis in two different cases, namely $s = \pm 1$.

3.1 Case I: Fermion as a wave function

If we choose $s = 1$ we have the case considered in Ref. [4] when the fermion is treated as a wave function. From the expression (21) we have the pressure

$$p_f = (2n - 1) V(\bar{\psi}, \psi). \quad (22)$$

According to (15), in regimes for which the mass term $m\bar{\psi}\psi$ in (15) may be neglected when compared with $V(\bar{\psi}, \psi)$ we have $p_f \approx (-1 + 2n)\rho_f$, with an equation of state parameter $w_f = -1 + 2n$, which corresponds to a XCDM model when

$n \neq 0$. For $n = 0$ we have the usual equation of state for the vacuum. In general, the fermion will have a negative pressure for $n < \frac{1}{2}$, with a phantom behaviour for $n < 0$. Such regime must be avoided in order to not violate the weak energy condition (WEC), namely $(\rho + p) > 0$.

The conservation equation (16) for the massless case is:

$$\dot{\rho}_f + 6nH\rho_f = 0, \quad (23)$$

whose solution in terms of the scale factor a is:

$$\rho_f = \rho_{f,0} \left(\frac{a_0}{a} \right)^{6n}, \quad (24)$$

and the Friedmann equation that follows from (6) for a matter energy density plus fermionic field contribution (24) is:

$$H^2(z) = H_0^2 \left[\Omega_m(1+z)^3 + \Omega_f(1+z)^{6n} \right]. \quad (25)$$

Here, $H(z)$ and H_0 are the Hubble parameter at the redshift z and at present time. $\Omega_m \equiv \rho_{m,0}/\rho_{c,0}$ is the present day density parameter for the matter content, $\Omega_f \equiv \rho_{f,0}/\rho_{c,0}$ is the density parameter for the fermion field and $\rho_{c,0} = 3H_0^2/8\pi G$ is the critical density.

The last term in (25) is the contribution of the fermionic field when treated as a wave function. For $n = 0$ we have the ordinary vacuum or cosmological constant density contribution, namely $\Omega_f \equiv \Omega_\Lambda$. The free parameters will be constrained with observational data in next section.

3.2 Case II: Fermion as a classical field

The choice $s = -1$ corresponds to the case where the fermion is a classical field. So, the field ψ is a Grassmann variable. It is very different from the wave function picture used in recent works [4, 5, 22–26], where the result is dependent from the parameter n in the potential $V(\bar{\psi}, \psi)$. The independence on the n parameter bring new interesting cosmological consequences, as we will see below.

When ψ is a Grassmann variable the choice $s = -1$ in the Eq. (21) gives

$$p_f = -V(\bar{\psi}, \psi). \quad (26)$$

From (15) we have the very interesting result that, for a massless fermionic field, the equation of state satisfied by the field is always of vacuum or dark energy type, namely $p_f = -\rho_f$, regardless of the form of the potential.

For the general case of a massive field the energy density (15) can be written in terms of the constant bilinear term associated to the mass, $\rho_f^* = m\bar{\psi}\psi$, namely $\rho_f = \rho_f^* +$

$V(\bar{\psi}, \psi)$, and the conservation equation (16) is:

$$\dot{\rho}_f + 3H\rho_f^* = 0, \quad (27)$$

whose solution in terms of the scale factor a is:

$$\rho_f = 3\rho_f^* \ln \left(\frac{a_0}{a} \right) + \rho_C, \quad (28)$$

and ρ_C is a constant density parameter that must satisfy some boundary condition. For the present time, $a = a_0$, this constant energy density can be associated to the cosmological constant, thus we will take $\rho_C = \rho_\Lambda$.

The Friedmann equation that follows from (6) for a standard matter energy density plus fermionic field contribution (28) is:

$$H^2 = H_0^2 \left[\Omega_m(1+z)^3 + 3\Omega^* \ln(1+z) + \Omega_\Lambda \right], \quad (29)$$

where $\Omega_m \equiv \rho_{m,0}/\rho_{c,0}$ is the present day matter density parameter, $\Omega^* \equiv \rho_f^*/\rho_{c,0}$ and $\Omega_\Lambda \equiv \rho_\Lambda/\rho_{c,0}$. The last two terms represents an effective time varying dark energy density parameter, a specific kind of $w(z)$ CDM model, which correctly reproduces the cosmological constant term at $z = 0$. Such parameters will be constrained with observational data in next section.

4 Constraints from observational data

The analyses to constraint the free parameters of the previous models were implemented in Python language. The standard method is to sampling the likelihood $\mathcal{L} \propto e^{-\chi^2/2}$ through Monte Carlo Markov Chain analysis, where the χ^2 function for Pantheon data is given by:

$$\chi_{SN}^2 = [\mathbf{m}_{obs} - \mathbf{m}(z, s)]^T \mathbf{C}_{SN}^{-1} [\mathbf{m}_{obs} - \mathbf{m}(z, s)]. \quad (30)$$

The parameter vector is represented by $\mathbf{s}$, with $\mathbf{m}_{obs}$ and $\mathbf{m}$ representing the observed apparent magnitude vector and model apparent magnitude, respectively, and $\mathbf{C}$ is a covariance matrix containing systematic errors for supernovae data [31]. We have marginalized the absolute magnitude M_B of SNe Ia over the interval $M_B \in [-21, -18]$. We have used the 1048 Supernovae type Ia data from Pantheon compilation [8].

For the $H(z)$ data we have used the 32 model independent $H(z)$ measurements obtained with Cosmic Chronometers method, obtained from Table I of Moresco et al. [7]. Also, due to the fact that $H(z)$ measurements from galaxy age are strongly biased with several systematic effects [7, 32, 33], we have implemented the covariance matrix effects described in

[33], so that the χ^2 function for the $H(z)$ data is:

$$\chi_H^2 = [H_{obs,i} - H(z_i, \mathbf{s})]^T \mathbf{D}_{ij}^{-1} [H_{obs,j} - H(z_j, \mathbf{s})], \quad (31)$$

where $\mathbf{D}_{ij}^{-1}$ represents the covariance matrix.

For the analysis with BAO data we have used signature estimate from various sources, as indicated in Tables II and III of Ref. [34] and the method described there and in [35].

4.1 Case I

In order to study the Case I with the fermionic wave function as a candidate to describe the dark energy in the universe, we start by analysing the model with $H(z)$ and SN Ia – Pantheon

data. First, notice that Eq. (25) for the present time gives $1 = \Omega_m + \Omega_f$, so that Ω_f can be written as a function of Ω_m and we are left with a 3 free parameters model, namely $\mathbf{s} = [H_0, \Omega_m, n]$. The parameter n characterizes the deviation from the standard Λ CDM model.

The contours at 1σ and 2σ c.l. for the parameters H_0 , Ω_m and n for the separate set of data are shown in Fig. 1, with $H(z)$ (grey), SNe Ia – Pantheon (red) and the combined analysis (blue). The mean values of the parameters for the joint analysis of $H(z) + \text{SN Ia}$ is $n = -0.05^{+0.16}_{-0.17}$, $\Omega_m = 0.325^{+0.096}_{-0.10}$ and $H_0 = 68.2^{+7.5}_{-7.3}$ at 95% c.l. In this first analysis no prior over n was imposed, and we can see that a negative value is favoured, which indicates a violation of the weak energy condition. Also, it is well known that $H(z)$ and Supernovae data does not constraint the value of the baryon

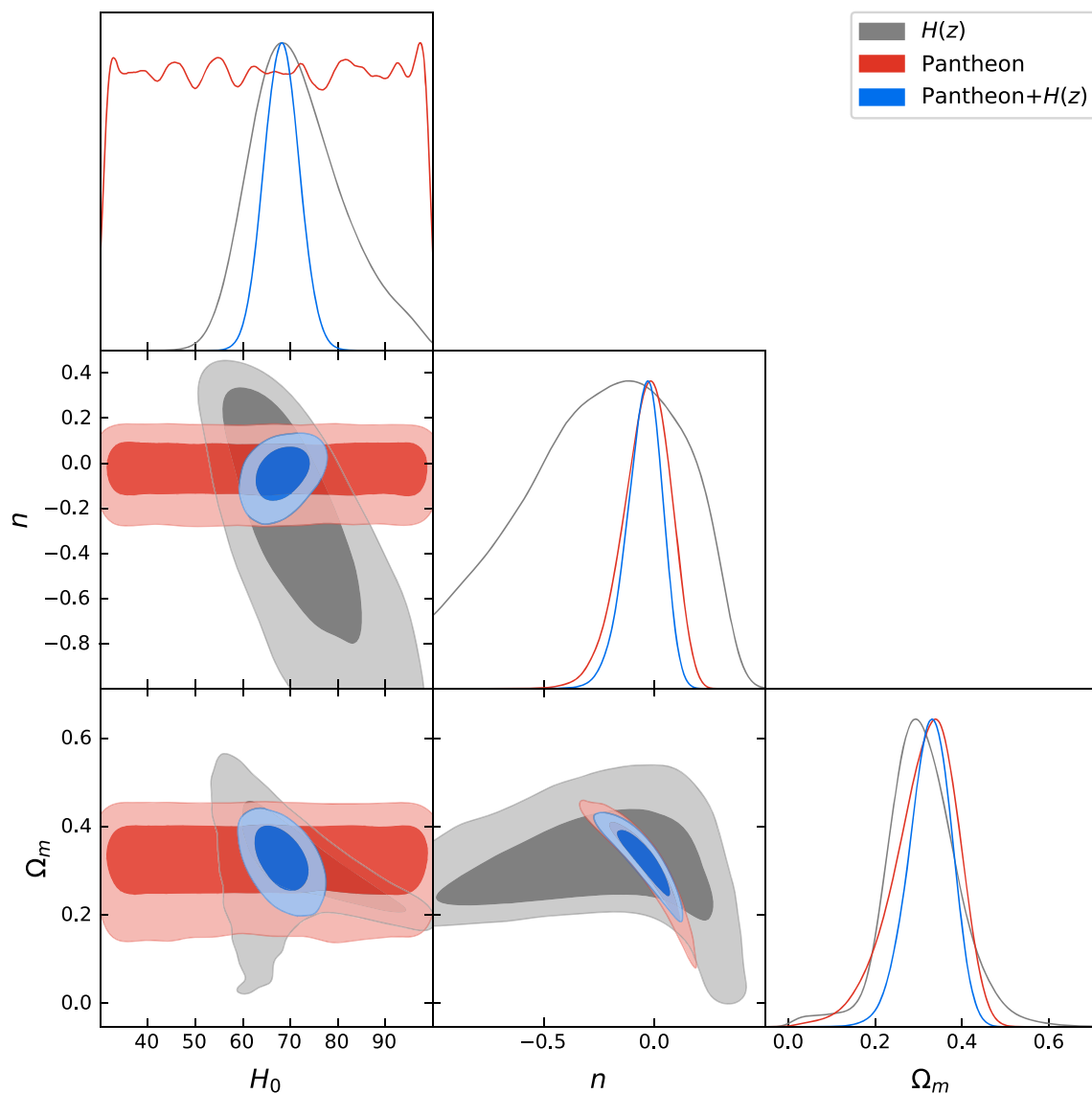


Fig. 1 Case I: Contours of the parameters H_0 , Ω_m and n at 1σ and 2σ c.l. for the separate set of data of $H(z)$ (grey) and SNe Ia – Pantheon (red), and for the joint analysis (blue)

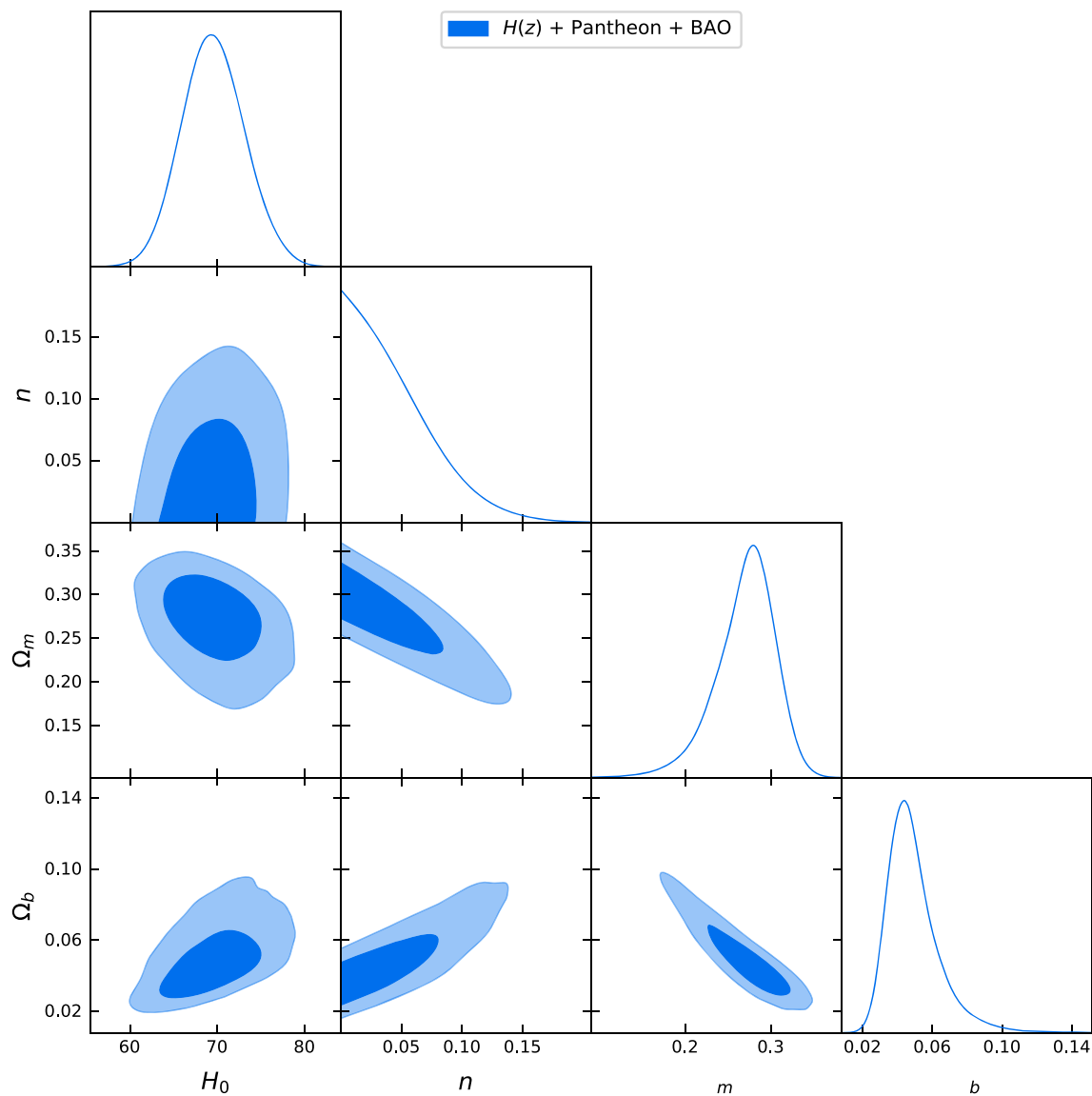


Fig. 2 Case I: Contours of the parameters H_0 , Ω_m , Ω_b and n at 1σ and 2σ c.l. for the joint analysis of $H(z)$ + SNe Ia (Pantheon) + BAO, with the prior $n \geq 0$

density parameter, Ω_b . Thus, in a second analysis we add the BAO data set and put a prior on the n parameter, namely $n \geq 0$. The contours at 1σ and 2σ c.l. are displayed in Fig. 2 for the joint analysis of $H(z)$ + SNe Ia (Pantheon) + BAO. The mean values and 95% c.l. limits for each parameter are displayed in Table 1. We can see that the values of the parameters H_0 and $\Omega_m + \Omega_b$ are in full agreement to the values obtained for the Λ CDM based model from Planck 2018 latest results [9] ($H_0 = (67.4 \pm 0.5) \text{ km/s/Mpc}$ and $\Omega_m = 0.315 \pm 0.007$ at 68% c.l.). The small positive value of the n parameter indicates the deviation from the standard Λ CDM model, which corresponds to $n = 0$. In next section we study the linear evolution of density perturbations for this model.

Table 1 Case I: Mean value of the parameters H_0 ($\text{km s}^{-1} \text{ Mpc}^{-1}$), Ω_m , Ω_b and n from Fig. 2 and 68%, 95% and 99% c.l. limits

Parameter	Mean value
H_0	$69.5^{+3.6+7.7+10}_{-3.6-7.0-10}$
Ω_m	$0.270^{+0.039+0.066+0.086}_{-0.026-0.075-0.15}$
Ω_b	$0.0493^{+0.0083+0.033+0.089}_{-0.017-0.026-0.031}$
n	$0.0^{+0.056+0.115+0.168}_{-0.0-0.0-0.0}$

4.2 Case II

For this case, Eq. (29) gives $1 = \Omega_m + \Omega_\Lambda$ at present time, so that Ω_Λ can be written as a function of Ω_m and we are left with a 3 free parameters model, namely $\mathbf{s} = [H_0, \Omega_m, \Omega^*]$. The

parameter Ω^* characterizes the deviation from the standard Λ CDM model.

We start the analysis with $H(z)$ and SNe Ia – Pantheon data. The contours at 1σ and 2σ c.l. for the parameters H_0 , Ω_m and Ω^* are shown in Fig. 3, with $H(z)$ (grey), SNe Ia (red) and the joint analysis of $H(z) + \text{SNe Ia}$ (blue). The mean values of the parameter are $\Omega^* = -0.06^{+0.21}_{-0.21}$, $\Omega_m = 0.33^{+0.12}_{-0.10}$ and $H_0 = 68^{+8}_{-7}$ at 95% c.l. As in the first case, no prior over Ω^* was imposed initially, and we can see that a negative value is allowed for the observational data. However such parameter is directly related to the mass of the fermionic field through $\Omega_f^* = \rho_f^*/\rho_{c,0}$, with $\rho_f^* = m\bar{\psi}\psi$. Thus, in a second analysis we impose the prior $\Omega^* \geq 0$ and add the BAO data set. The contours at 1σ and 2σ c.l. are displayed in Fig. 4 for the joint analysis of $H(z) + \text{SNe Ia}$ (Pantheon) +

BAO. The mean values and 95% c.l. limits for each parameter are displayed in Table 2. The values of the parameters H_0 and $\Omega_m + \Omega_b$ are in full agreement to the values obtained for the Λ CDM based model from Planck 2018 results [9]. The small positive value of the Ω^* parameter indicates the deviation from the standard Λ CDM model, which corresponds to $\Omega^* = 0$. In next section we study the linear evolution of density perturbations for this model.

5 Linear evolution of density perturbations

As a final analysis, let us study the linear evolution of density perturbations related to the models of Cases I and II, where a non null value of the parameters n and Ω^* represents the

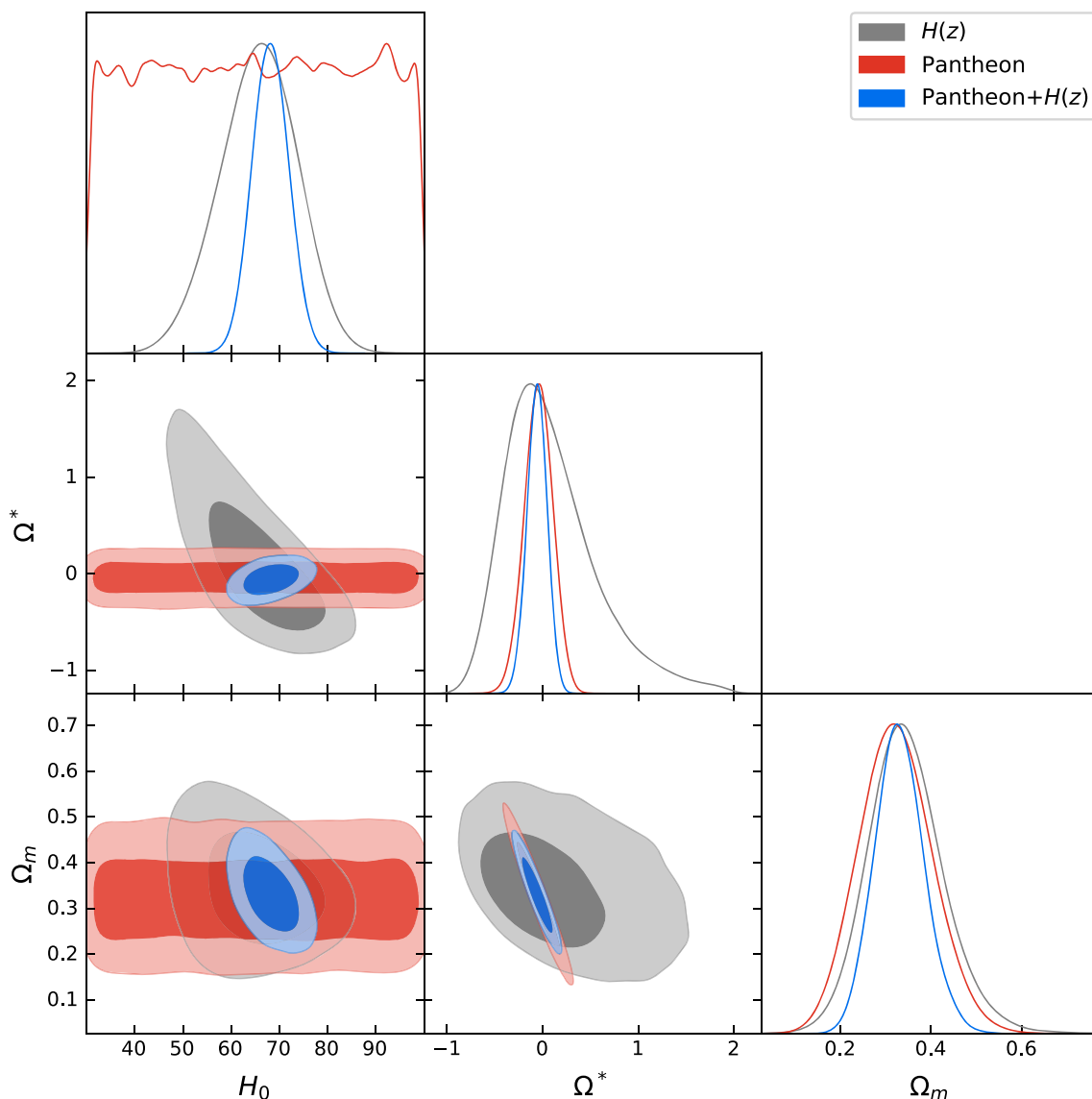


Fig. 3 Case II: Contours of the parameters H_0 , Ω_m and Ω^* at 1σ and 2σ c.l. for the separate set of data of $H(z)$ (grey), SNe Ia – Pantheon (red) and the joint analysis (blue)

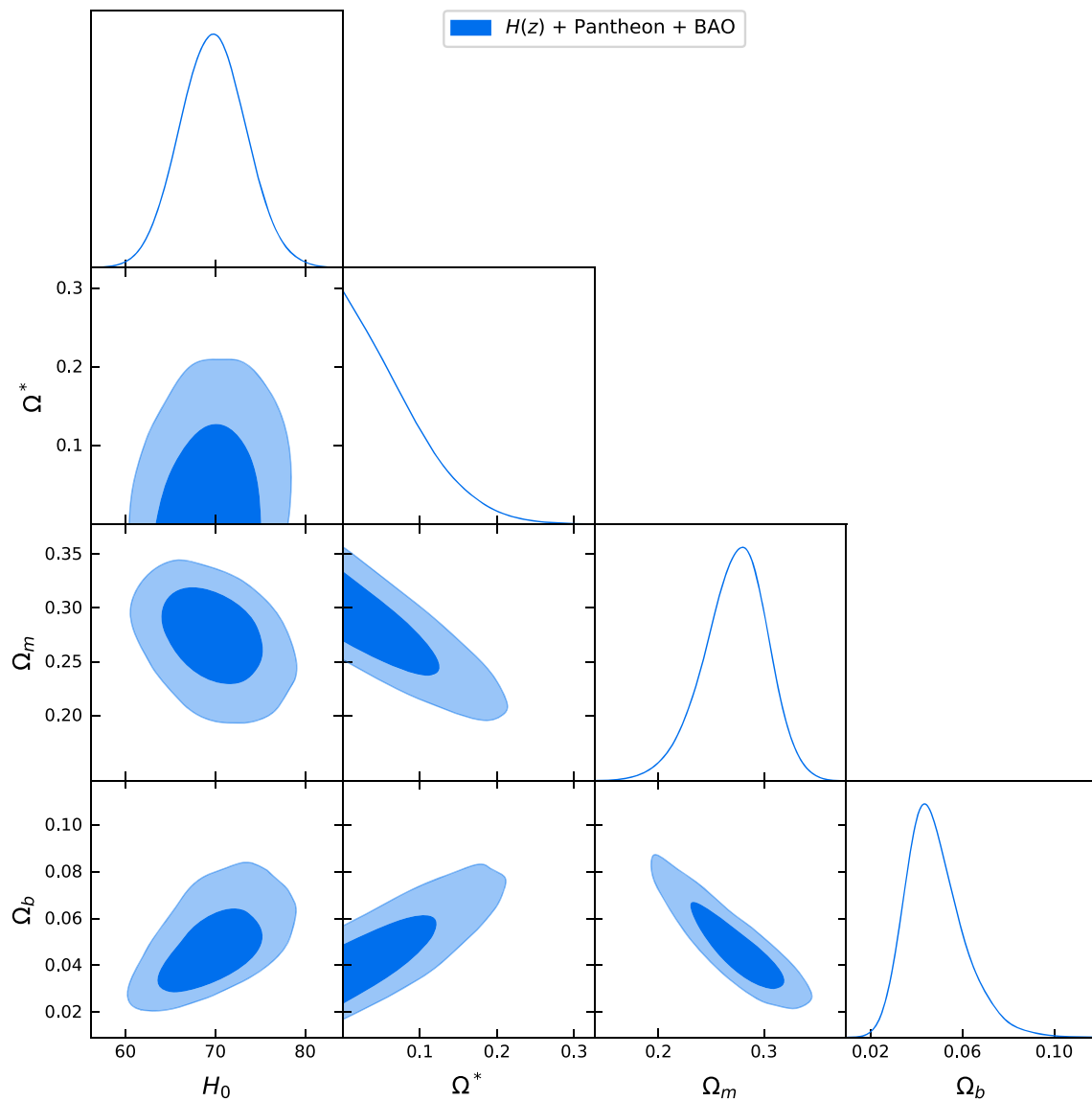


Fig. 4 Case II: Contours of the parameters H_0 , Ω_m , Ω_b and Ω^* at 1σ and 2σ c.l. for the joint analysis of $H(z)$ + SNe Ia (Pantheon) + BAO, with the prior $\Omega^* \geq 0$

Table 2 Case II: Mean value of the parameters H_0 ($\text{km s}^{-1} \text{Mpc}^{-1}$), Ω_m , Ω_b and Ω^* from Fig. 4 and 68%, 95% and 99% c.l. limits

Parameter	Mean value
H_0	$69.7^{+3.7+7.4+10.0}_{-3.7-7.2-10.0}$
Ω_m	$0.272^{+0.033+0.056+0.080}_{-0.026-0.068-0.11}$
Ω_b	$0.048^{+0.009+0.027+0.057}_{-0.015-0.023-0.029}$
Ω^*	$0.0^{+0.084+0.176+0.262}_{-0.0-0.0-0.0}$

deviation from Λ CDM model. For both cases we have an effective equation of state parameter w_f whose effects on the primordial density perturbations can be studied using the techniques developed by Abramo et al. [36], where structure

formation in the presence of dark energy perturbations are considered in the so called pseudo-Newtonian approach.

Written the conservation equation (16) as:

$$\frac{d\rho_f}{dz} = \frac{3(1+w_f)}{(1+z)}\rho_f, \quad (32)$$

and using (24) and (28) for Cases I and II, respectively, we obtain:

$$w_f = -1 + 2n, \quad \text{Case I} \quad (33)$$

$$w_f = -1 + \frac{\Omega^*}{\Omega_f(z)}, \quad \text{Case II} \quad (34)$$

where $\Omega_f(z) = 3\Omega^* \ln(1+z) + \Omega_\Lambda$ represents an effective dark energy density parameter which evolves with time,

according to (29). At this point it becomes clear that Case I represents a Λ CDM model with a constant equation of state parameter and Case II represents a $w(z)$ CDM model, with a varying equation of state parameter.

The cosmological perturbations can be introduced by admitting inhomogeneous deviations from the background quantities,

$$\begin{aligned}\rho_m &= \bar{\rho}_m(1 + \delta_m), & p_m &= \bar{p}_m + \delta p_m, \\ \rho_f &= \bar{\rho}_f(1 + \delta_f), & p_f &= \bar{p}_f + \delta p_f,\end{aligned}\quad (35)$$

and therefore $\delta_m = \delta\rho_m(\vec{x}, t)/\bar{\rho}_m$ and $\delta_f = \delta\rho_f(\vec{x}, t)/\bar{\rho}_f$ are the density contrasts for each fluid. The linear regime is described by the coupled differential equations (see Eqs. (29) and (30) of [36]):

$$\ddot{\delta}_m + 2H\dot{\delta}_m = \frac{3H^2}{2}[\Omega_m(t)\delta_m + \Omega_f(t)\delta_f(1 + 3w_f)], \quad (36)$$

$$\begin{aligned}\ddot{\delta}_f + \left(2H - \frac{\dot{w}_f}{1 + w_f}\right)\dot{\delta}_f \\ = \frac{3H^2}{2}(1 + w_f)[\Omega_m(t)\delta_m + \Omega_f(t)\delta_f(1 + 3w_f)],\end{aligned}\quad (37)$$

where a dot stands for time derivative and the density parameters $\Omega_m(t) = \Omega_m(a(t))$ and $\Omega_f(t) = \Omega_f(a(t))$ are functions of time at this level, but can be rewritten as a function of the redshift in the standard manner, using $\frac{d}{dt} = -(1+z)H(z)\frac{d}{dz}$.

The solution of the system of coupled differential equations (36)–(37) can be obtained numerically with the initial conditions at $z_i = 10^3$, namely $\delta_m(z_i) = 1/(1 + z_i)$ and $\delta'_m(z_i) = -\delta_m(z_i)/(1 + z_i)$, where a prime denotes derivative with respect to the redshift. The initial conditions of the dark energy density are assumed to be null.

In Fig. 5 we show the effects of fermionic field fluctuations on the growth of dark matter perturbation δ_m for both cases with different values of n and Ω^* . The Λ CDM model is also present for comparison. On the top panel we see that the effect of the dark energy perturbation represented by the fermionic field is to diminish the growth of the matter perturbation for late time. For the best fit values of n and Ω^* , green and red curves respectively, the growth of matter perturbation are quite indistinguishable each other. For 2σ the models are slightly different, yellow and blue curves, respectively.

The growth function $f(z)$ defined by $f = -(1+z)\frac{\delta'_m}{\delta_m}$ is showed in the bottom panel. The main behaviours are quit similar to the growth of density contrast. Although being indistinguishable from Λ CDM model (black line) at early time, the fermionic model predicts the formation of fewer structures over the evolution for late time, as expected when compared to standard model. Such qualitative behaviour could alleviate at least one of the so called “small scale problems” [37] associated to Λ CDM model, which predict too many dwarf galaxies and too much dark matter in the inner-

most regions of galaxies. The presence of an overdensity of dark energy in the same regions of dark matter density perturbations could diminish the net effect of dark matter.

6 Conclusion

In this paper we have investigated the late time cosmological evolution driven by a fermionic field endowed with a potential written in terms of scalar and pseudoscalar invariants. Two slightly different scenarios are possible, namely, whether the fermion field is considered a standard wave-function or a Grassmannian variable. In both approaches the energy density and pressure of the field can be treated as effective thermodynamic quantities that enters the Friedmann evolution equation satisfying an equation of state of the form $p_f \sim w\rho_f$, with w an effective equation of state parameter.

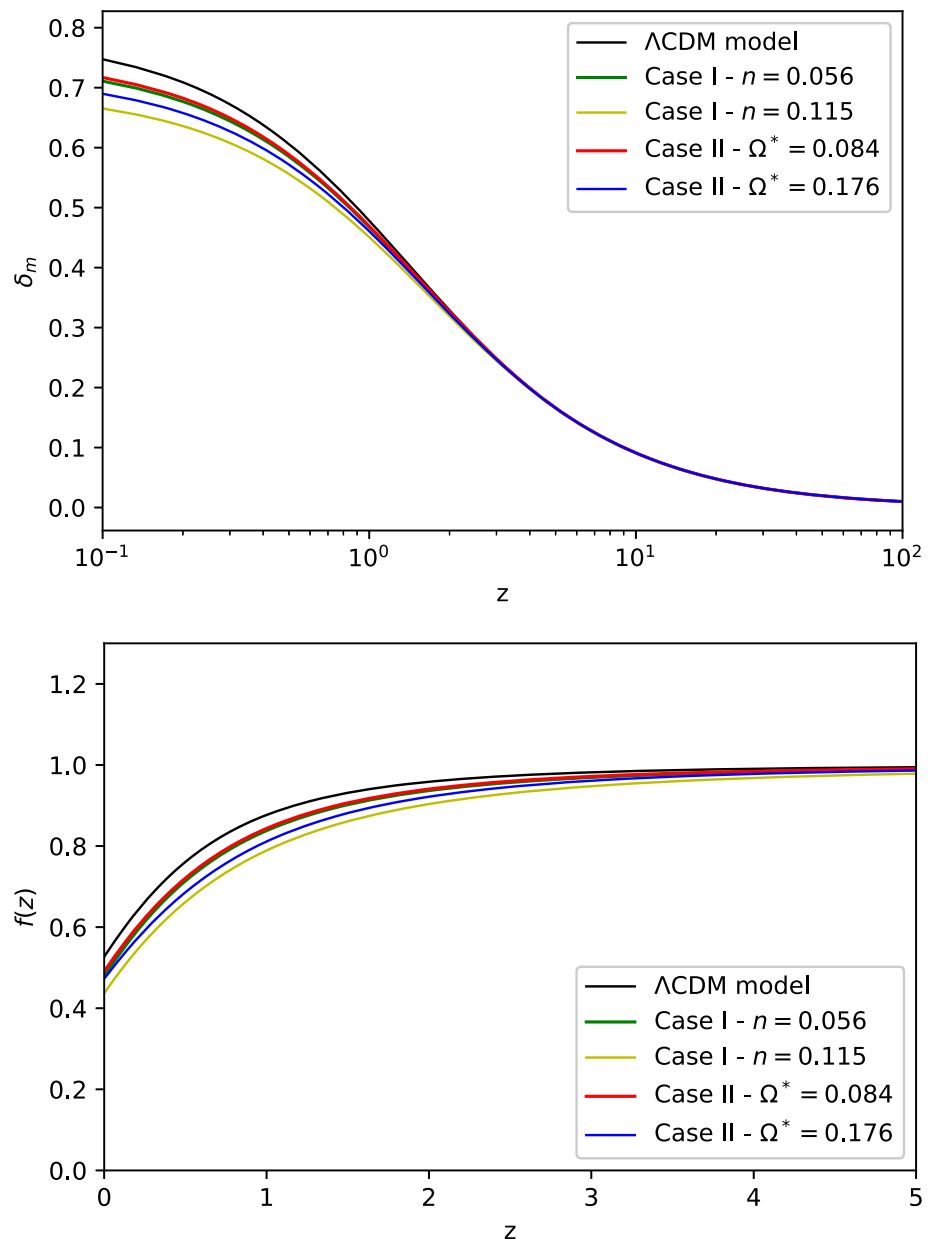
The free parameters of the model composed by a total matter energy density ρ_m and fermion field energy density ρ_f were constrained with Supernovae and Hubble parameter observational data. The inclusion of BAO data into the analysis allows the constraint of baryonic density parameter separate out of the total matter density. The H_0 parameter was also set as a free parameter in order to compare with the recent Planck 2018 results.

In the first approach, where a massless fermionic field is treated as a wave-function, the equation of state can be written as $p_f = (-1 + 2n)\rho_f$. Such model is equivalent to a Λ CDM model with constant equation of state parameter. The free parameter n , which represents the deviation of the model from the Λ CDM model, was constrained by observational data, furnishing $n < 0.115$ at 2σ c.l. The values of H_0 , Ω_m and Ω_m (see Table 1) are also in good agreement to latest Planck 2018 results [9].

In the second approach, where a massive fermionic field is treated as a Grassmannian variable, the equation of state can be written as $p_f = -\rho_f + \rho_f^*$, where the term $\rho_f^* = m\bar{\psi}\psi$ represents a deviation from a vacuum equation of state parameter, depending on the mass m of the fermionic field. The model is equivalent to a $w(z)$ CDM model. For $m = 0$ the system represents the Λ CDM model. The constraint with observational data gives the value $\Omega^* < 0.176$ at 2σ c.l. The values of the parameters H_0 , Ω_m and Ω_b (see Table 2) are also in good agreement to latest Planck 2018 results [9].

Both analyzes show that the fermionic field with a potential written in terms of bilinear invariants of the form (18) are in good agreement with the latest Planck 2018 results for the Λ CDM model, representing an extension of the standard model and given a new interpretation for the dark energy component. For the specific case of the second approach, where the fermionic field is a Grassmann variable, a much

Fig. 5 Top panel: Growth of the density contrast δ_m of matter for both cases. The curves were plotted with the values of the parameters in Tables 1 and 2, with different values of n and Ω^* . The evolution of matter density contrast for Λ CDM model is also presented for comparison (upper black line). Bottom panel: Growth function f of matter for the fermionic model and for Λ CDM model (black line) for comparison



more interesting feature appears. If we neglect the mass m of the fermion in (15), Eq. (26) ensures that the fermion behaves like the vacuum energy for any potential of the form (18). This shows that the intrinsic Grassmannian property of the field leads always to a dark energy or negative pressure contribution on the evolution, irrespective the specific form of the potential. From a quantum point of view this feature may be related to the degeneracy pressure intrinsic to the genuine fermion field.

Finally, the study of linear evolution of density perturbations for both cases show that, at least qualitatively, the problem of overdensities predicted by Λ CDM model during the formation of the first structures can be alleviated for non negative values of the parameters n and Ω^* , since that for both cases the growth function f are smaller than for

the Λ CDM model. Additionally, as extensions of the standard model, further investigations on the H_0 tension and σ_8 problem [38–41] must be carried out for both cases.

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Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The data used in the manuscript are freely available from Table I of reference [7] for $H(z)$ data and from [8] for Pantheon compilation, for this reason they will not be deposited.]

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